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First-order compatibility for a (2 + 1)-dimensional diffusion equation

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Abstract
A class of first-order partial differential equations is obtained compatible with a (2 + 1)-dimensional linear diffusion equation with a nonlinear source term. We will show that if the source term is quasilinear, then compatible equations are quasilinear. Furthermore, we obtain diffusion equations with source terms that admit non-quasilinear compatible equations.

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1. Introduction

Symmetry analysis has played a fundamental role in the construction of exact solutions to nonlinear partial differential equations. Based on the original work of Lie [17] on continuous groups, symmetry analysis provides a unified explanation for the seemingly diverse and ad hoc integration methods used to solve ordinary differential equations. For equations in (1 + 1) dimensions, one seeks the invariance of a differential equation

$$\Omega(t, x, u, u_t, u_x, u_{tt}, u_{tx}, \ldots) = 0, \tag{1.1}$$

under the group of infinitesimal transformations

$$\begin{aligned} \bar{t} &= t + T(t, x, u)\epsilon + O(\epsilon^2), \\ \bar{x} &= x + X(t, x, u)\epsilon + O(\epsilon^2), \\ \bar{u} &= u + U(t, x, u)\epsilon + O(\epsilon^2). \end{aligned} \tag{1.2}$$

This leads to a set of determining equations for the infinitesimals T, X and U which, when solved, gives rise to the symmetries of (1.1). Once a symmetry is known for a differential equation, invariance of the solution leads to the invariant surface condition

$$Tu_t + Xu_x = U. \tag{1.3}$$

Solutions of (1.3) lead to a solution ansatz, which substituted into equation (1.1) leads to a reduction of the original equation. A generalization of the so-called classical method of Lie was proposed by Bluman and Cole [3], which today is commonly referred to as the 'nonclassical method'. Their method seeks invariance of the original equation (1.1) augmented with the invariant surface condition (1.3). At the present time, there is extensive literature on the subject and we refer the reader to the books by Bluman and Kumei [5], Olver [22] and Rogers and Ames [27].

A particular class of partial differential equations that has benefited tremendously from this type of analysis are nonlinear diffusion equations. For example, the nonlinear diffusion–convection equation

$$u_t = \nabla \cdot (D(u)\nabla u) - K'(u)u_z \tag{1.4}$$

has a variety of applications to porous media, including displacement of one liquid by another (Fokas and Yortsos [10]), unsaturated flow (Klute [16]) and the transport of a solute with absorption to pore surfaces (Rosen [28]). In the context of hydrological flows, $D(u)$ is the concentration-dependent diffusivity and based on Darcy's law (Klute [16]), $K(u)$ is the concentration-dependent hydraulic conductivity. It has been used by Clothier *et al* [7] in the case of $D(u) = \text{const.}$ and $K(u) = \text{quadratic}$ for unsaturated flows in field soils and after suitable translation and scaling (1.4) becomes

$$u_t = \nabla^2 u - u u_z, \tag{1.5}$$

which is known as Burgers' equation. In one space dimension (1.5) is known to be linearizable; however, this does not extend to higher dimensions [6].

A second example is the model

$$n_t = D\nabla^2 n + \Lambda |\nabla n|^2 + \lambda n G(n, n_m) \tag{1.6}$$

introduced by Grimson and Barker [15] for the spatiotemporal growth of bacterial colonies with local and nonlocal modes of growth. Here, n is the local microbial number density, D is the diffusion coefficient, and Λ and λ are the constants. The presence of the nonlinear terms in (1.6) represents (i) the local growth of cell number up to a maximum of n_m (the third term on the right-hand side of (1.6)) and (ii) the nonlocal growth occuring at concentration gradients required in systems where the diffusion constant is sufficiently small with respect to the local growth so that the colony cannot spread by diffusion alone (the second term on the right-hand side of (1.6)). Further examples can be found in the evolution of grain boundaries [30] and image processing [24] but the diffusion there tends to be nonlinear.

Symmetry properties of equations such as (1.4) and (1.6) have been considered by a number of authors. Two- and three-dimensional reaction–diffusion equations of the form

$$u_t = \nabla \cdot (D(u)\nabla u) + Q(u) \tag{1.7}$$

were considered by Dorodnitsyn *et al* [8] and by Galaktionov *et al* [12] who gave a list of diffusion and source terms admitting a classical symmetry. Nonclassically, equation (1.7) was first considered by Goard and Broadbridge [14] in two spatial dimensions who showed that nonclassical symmetries exist. These were further exploited by Gandarias and del Aguila [13] who provide many reductions of (1.7) in the case of $D = 1$. In the case of higher dimensional diffusion equations with convection, i.e. equation (1.4), a classical analysis was performed by Edwards and Broadbridge [9]. We also note the symmetry analysis performed by Bindu *et al* [4] on

$$u_t = u_{xx} + u_{yy} + \frac{m}{1-u}\left(u_x^2 + u_y^2\right) + u(1-u), \tag{1.8}$$

which led to its linearization in the case of $m = 2$. Recently, a symmetry analysis of the equation

$$u_t = u_{xx} + u_{yy} + Q(u, u_x, u_y) \tag{1.9}$$

was conducted by Arrigo, Suazo and Sule [2], where it was shown that a variety of source terms exist that admit a nontrivial symmetry (those symmetries not obtained by inspection).

While both the classical and nonclassical symmetry methods have had tremendous success when applied to a wide variety of physically important nonlinear differential equations, there exist exact solutions to partial differential equations that cannot be explained using symmetry analysis. For example, Galaktionov [11] showed that the PDE

$$u_t = u_{xx} + u_x^2 + u^2 \tag{1.10}$$

admits the solution

$$u = a(t)\cos x + b(t), \tag{1.11}$$

where $a(t)$ and $b(t)$ satisfy the system of ODEs:

$$\dot{a} = -a + 2ab, \qquad \dot{b} = a^2 + b^2. \tag{1.12}$$

A classical symmetry analysis of equation (1.10) leads to only translational symmetries in space and time. The associated invariant surface condition is

$$u_t + cu_x = 0, \tag{1.13}$$

where c is a constant and (1.13) will clearly not give rise to solution (1.11). A nonclassical symmetry analysis with $T = 1$ gives rise to the same invariant surface condition while with $T = 0$ gives rise to solving the original equation (see, for example, Zhdanov and Lahno [31]).

However, Olver [23] was able to show that the solution obtained by Galaktionov can be obtained by the method of differential constraints. By appending (1.10) with

$$u_{xx} - \cot x u_x = 0, \tag{1.14}$$

he showed that (1.11) could be obtained.

Differential constraints or compatibility of partial differential equations have been around for quite some time and date back to the pioneering work of Riquier (1893) and Cartan (1901) (see Pommariet [25] for further details and historical references) and have been successfully applied to equations in fluid mechanics (see Meleshko [19] and the references within) and second-order evolution equations in (1+1) dimensions (see, for example Olver [23]). However, only recently has the connection been made with the nonclassical method of symmetry analysis (see Pucci and Saccomandi [26], Seiler [29], Arrigo and Beckham [1], Nui and Pan [20], and Nui, Huang and Pan [21]).

In this paper, we consider the compatibility between the (2 + 1) dimensional reaction—diffusion equation

$$u_t = u_{xx} + u_{yy} + Q(u, u_x, u_y), \tag{1.15}$$

and the first-order partial differential equation

$$u_t = F(t, x, y, u, u_x, u_y), \tag{1.16}$$

in which we prove the following.

Theorem. *Every equation which is equivalent to equation (1.15) with a quasilinear source term Q is compatible with (1.16) if F is quasilinear and (ii) imposing the condition of non-*

quasilinearity

$$(F_{pp}, F_{pq}, F_{qq}) \neq (0, 0, 0), \tag{1.17}$$

every equation which is equivalent to (1.15) with a source term of the form

$$Q = cu + G(u_x, u_y),$$

is compatible with equations of the form

$$u_t = cu + G(u_x, u_y),$$

where c is an arbitrary constant and $G(p, q)$ is a function satisfying $G_{pp} + G_{qq} = 0$.

2. First-order compatibility

Compatibility between (1.15) and (1.16) gives rise to the compatibility equation constraints

$$F_{pp} + F_{qq} = 0, \tag{2.1a}$$

$$F_{xp} - F_{yq} + pF_{up} - qF_{uq} + (F - Q)F_{pp} = 0, \tag{2.1b}$$

$$F_{xq} + F_{yp} + qF_{up} + pF_{uq} + (F - Q)F_{pq} = 0, \tag{2.1c}$$

$$\begin{aligned} -F_t + F_{xx} + F_{yy} + 2pF_{xu} + 2qF_{yu} + 2(F - Q)F_{yq} + (p^2 + q^2)F_{uu} + 2q(F - Q)F_{uq} \\ + (F - Q)^2 F_{qq} + Q_p F_x + Q_q F_y + (pQ_p + qQ_q - Q)F_u \\ - pQ_u F_p - qQ_u F_q + FQ_u = 0. \end{aligned} \tag{2.1d}$$

Eliminating the x and y derivatives in (2.1b) and (2.1c) by (i) cross differentiation and (ii) imposing (2.1a) gives

$$2F_{up} + (F_p - Q_p)F_{pp} + (F_q - Q_q)F_{pq} = 0, \tag{2.2a}$$

$$2F_{uq} + (F_p - Q_p)F_{pq} + (F_q - Q_q)F_{qq} = 0. \tag{2.2b}$$

Further, eliminating F_{up} and F_{uq} by again (i) cross differentiation and (ii) imposing (2.1a) gives rise to

$$(2F_{pp} - Q_{pp} + Q_{qq})F_{pp} + 2(F_{pq} - Q_{pq})F_{pq} = 0, \tag{2.3a}$$

$$(Q_{pp} - Q_{qq})F_{pq} + 2Q_{pq}F_{qq} = 0. \tag{2.3b}$$

Solving (2.1a), (2.3a) and (2.3b) for F_{pp}, F_{pq} and F_{qq} gives rise to two cases:

$$\text{(i)}\, F_{pp} = F_{pq} = F_{qq} = 0, \tag{2.4a}$$

$$\text{(ii)}\, F_{pp} = \tfrac{1}{2}(Q_{pp} - Q_{qq}), \qquad F_{pq} = Q_{pq}, \qquad F_{qq} = \tfrac{1}{2}(Q_{qq} - Q_{pp}). \tag{2.4b}$$

As we are primarily interested in compatible equations that are more general than quasilinear, we omit the first case. In the second case, we see that if Q is of the form

$$Q = f_0(u)(p^2 + q^2) + f_1(u)p + f_2(u)q + f_3(u), \tag{2.5}$$

for arbitrary functions $f_0 - f_3$, then

$$Q_{pq} = 0, \qquad Q_{pp} - Q_{qq} = 0,$$

identically and from (2.4b) we obtain

$$F_{pp} = F_{pq} = F_{qq} = 0.$$

Thus, for source terms of form (2.5), we only have quasilinear compatible equations. Furthermore, since equations with this type of source term are equivalent to equations with source terms with $f_0 = 0$ (equivalent in the sense that we can transform between equations with a suitable transformation $u = \phi(\tilde{u})$), we can set $f_0 = 0$ without loss of generality. Thus, for quasilinear source terms, F is linear in p and q giving our first result. Hereinafter, we will impose the non-quasilinearity condition (1.17).

If we require that the three equations in (2.4b) be compatible, then to within equivalence transformations of the original equation, Q satisfies

$$Q_{pp} + Q_{qq} = 0. \tag{2.6}$$

Using (2.6), we find that (2.4b) becomes

$$F_{pp} = Q_{pp}, \qquad F_{pq} = Q_{pq}, \qquad F_{qq} = Q_{qq}, \tag{2.7}$$

from which we find that

$$F = Q(u, p, q) + X(t, x, y, u)p + Y(t, x, y, u)q + U(t, x, y, u), \tag{2.8}$$

where X, Y and U are arbitrary functions. Substituting (2.8) into (2.2a) and (2.2b) gives

$$2Q_{up} + XQ_{pp} + YQ_{pq} + 2X_u = 0, \tag{2.9a}$$

$$2Q_{uq} + XQ_{pq} + YQ_{qq} + 2Y_u = 0, \tag{2.9b}$$

while (2.1b) and (2.1c) become (using (2.6) and (2.9))

$$(Xp + Yq + 2U)Q_{pp} + (Xq - Yp)Q_{pq} + 2(X_x - Y_y) = 0, \tag{2.10a}$$

$$(Xq - Yp)Q_{pp} - (Xp + Yq + 2U)Q_{pq} - 2(X_y + Y_x) = 0. \tag{2.10b}$$

If we differentiate (2.9a) and (2.9b) with respect to x and y, we obtain

$$X_xQ_{pp} + Y_xQ_{pq} + 2X_{xu} = 0, \qquad X_xQ_{pq} + Y_xQ_{qq} + 2Y_{xu} = 0, \tag{2.11a}$$

$$X_yQ_{pp} + Y_yQ_{pq} + 2X_{yu} = 0, \qquad X_yQ_{pq} + Y_yQ_{qq} + 2Y_{yu} = 0. \tag{2.11b}$$

If $X_x^2 + Y_x^2 \neq 0$, then solving (2.6) and (2.11a) for Q_{pp}, Q_{pq} and Q_{qq} gives

$$Q_{pp} = -Q_{qq} = \frac{2(Y_xY_{xu} - X_xX_{xu})}{X_x^2 + Y_x^2}, \qquad Q_{pq} = -\frac{2(X_xY_{xu} + Y_xX_{xu})}{X_x^2 + Y_x^2}.$$

If $X_y^2 + Y_y^2 \neq 0$, then solving (2.6) and (2.11b) for Q_{pp}, Q_{pq} and Q_{qq} gives

$$Q_{pp} = -Q_{qq} = \frac{2(Y_yY_{yu} - X_yX_{yu})}{X_y^2 + Y_y^2}, \qquad Q_{pq} = -\frac{2(X_yY_{yu} + Y_yX_{yu})}{X_y^2 + Y_y^2}.$$

In any case, this shows that Q_{pp}, Q_{pq} and Q_{qq} are at most functions of u only. Thus, if we let

$$Q_{pp} = -Q_{qq} = 2g_1(u), \qquad Q_{pq} = g_2(u),$$

for arbitrary functions g_1 and g_2, then Q has the form

$$Q = g_1(u)(p^2 - q^2) + g_2(u)pq + g_3(u)p + g_4(u)q + g_5(u), \tag{2.12}$$

where $g_3 - g_5$ are further arbitrary functions. Substituting (2.12) into (2.10) gives

$$2(Xp + Yq + 2U)g_1 + (Xq - Yp)g_2 + 2(X_x - Y_y) = 0, \tag{2.13a}$$

$$2(Xq - Yp)g_1 - (Xp + Yq + 2U)g_2 - 2(X_y + Y_x) = 0. \tag{2.13b}$$

Since both equations in (2.13) must be satisfied for all p and q, this requires that each coefficient of p and q must vanish. This leads to

$$2g_1X - g_2Y = 0, \qquad g_2X + 2g_1Y = 0, \tag{2.14a}$$

$$2g_1U + X_x - Y_y = 0, \qquad g_2U + X_y + Y_x = 0. \tag{2.14b}$$

From (2.14a) we see that either $g_1 = g_2 = 0$ or $X = Y = 0$. If $g_1 = g_2 = 0$, then Q is quasilinear giving that F is quasilinear which violates our non-quasilinearity condition (1.17). If $X = Y = 0$, we are led to a contradiction as we imposed $X_x^2 + Y_x^2 \neq 0$ or $X_y^2 + Y_y^2 \neq 0$. Thus, it follows that

$$X_x^2 + Y_x^2 = 0, \qquad X_y^2 + Y_y^2 = 0,$$

or

$$X_x = 0, \qquad X_y = 0, \qquad Y_x = 0, \qquad Y_y = 0.$$

Since Q is not quasilinear then from (2.10) we deduce that

$$(Xp + Yq + 2U)^2 + (Xq - Yp)^2 = 0, \tag{2.15}$$

from which we obtain $X = Y = U = 0$. With this assignment we see from (2.8) that $F = Q$ and from (2.9) that Q satisfies

$$Q_{up} = 0, \qquad Q_{uq} = 0, \tag{2.16}$$

which has the solution

$$Q = G(p, q) + H(u), \tag{2.17}$$

for arbitrary functions G and H. From (2.6), we find that G satisfies $G_{pp} + G_{qq} = 0$ while from (2.1d), that H satisfies $H'' = 0$ giving that $H = cu$ where c is an arbitrary constant noting that we have suppressed the second constant of integration due to translational freedom. This leads to our main result. Equations of the form

$$u_t = u_{xx} + u_{yy} + cu + G(u_x, u_y),$$

are compatible with the first-order equations

$$u_t = cu + G(u_x, u_y).$$

3. Conclusion

In this paper, we have considered the compatibility between a diffusion equation with a source term in (2 + 1) dimensions and general first-order partial differential equations. We have shown that if the source term is quasilinear then compatible equations are quasilinear while imposing the condition that compatible equations are fully nonlinear gives rise to a large class of compatible reaction–diffusion equations.

One naturally asks about compatibility with the diffusion equations in three dimensions. We have considered the compatibility between

$$u_t = u_{xx} + u_{yy} + u_{zz} + Q(u, u_x, u_y, u_z),$$

and

$$u_t = F(t, x, y, z, u, u_x, u_y, u_z),$$

and have found that

$$F_{pp} = F_{pq} = F_{pr} = F_{qq} = F_{qr} = F_{rr} = 0,$$

where $p = u_x$, $q = u_y$ and $r = u_z$ which gives that F is linear in p, q and r regardless of Q.

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